International Journal of Theoretical Physics, Vol. 43, No. 6, June 2004 (© 2004)

Dirac on Gauges and Constraints

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Received

We examine the relevance of Dirac's view on the use of transformation theory and invariants in modern physics to current reflections on the meaning of physical symmetries, especially gauge symmetries.

KEY WORDS: Dirac; transformation theory; constrained systems; gauge symmetries.

1. INTRODUCTION

In his preface to the first edition of his masterpiece *The Principles of Quantum Mechanics* in 1930 P. A. M. Dirac wrote

> It has become increasingly evident in recent times . . . that [nature's] fundamental laws control a substratum of which we cannot form a mental picture without introducing irrelevancies. The formulation of these laws requires the use of the mathematics of transformations. The important things in the world appear as the invariants . . . of these transformations (Dirac, 1987)

and proceeded by adding

> The growth of the use of transformation theory, as applied first to relativity and later to the quantum theory, is the essence of the new method in theoretical physics. Further progress lies in the direction of making our equations invariant under wider and still wider transformations.

These were indeed prophetic words. As foreseen by Dirac, twentieth-century physical developments have been significantly guided by the search for larger invariance properties of physical laws. The notions of *invariance*, *transformation group*, and *symmetry* ("invariance under a transformation group") have notably acquired a crucial role in the field. If physics has much progressed since the first appearance of Dirac's book on quantum mechanics, it is in large part due to the role of symmetry principles and their group-theoretical exploitation. This is well-known

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0020-7748/04/0600-1503/0 © 2004 Springer Science+Business Media, Inc.

history, and physics textbooks now typically devote a good part to explaining how symmetries work in physical theories and how they lead to so many important physical consequences.

But there is also another aspect of Dirac's remarks that, as we want to show in this paper, is of special significance from today's perspective: that is, his *reflections* on the use of transformations and invariants in modern physics. The role and relevance of symmetries to physics, which are well known and uncontroversial, have in fact created an interpretative problem. Where do symmetries come from, what is their real meaning? After the reflections by Weyl (1982) and Wigner (1967)—the classic references on the subject—questions of this kind have received different sorts of answers in the literature. Symmetries are of different types ("external" or internal,' "global" or "local," "continuous" or "discrete,"...) and, accordingly, depending on the type of symmetry considered different options appear be favored. For example, spacetime (or "external") symmetries are generally taken to motivate realistic or ontic options (symmetries as part of the ontology or structure of the physical world); while other types of symmetries, especially the gauge (or "local") symmetries of the Standard Model, appear to provide good reasons for preferring options of an epistemic sort (the presence of symmetries is related, first of all, to our way of representing the physical world). In fact, whether it is possible to provide a satisfactory and unitary view on the meaning of physical symmetries remains an open question, at the center of an intensive debate among physicists and philosophers of physics.[2]

It is in the light of this discussion that, in this paper, we shall consider Dirac's view as it results from examining his 1930 preface and in general his writings. That is, the view that the application of transformation theory and the related use of invariants in theoretical physics is closely connected with the inevitable introduction of "irrelevancies" when attempting a fundamental description of the physical world. According to Dirac, while the classical world can be considered "to be an association of observable objects ... moving about according to definite laws of force, so that one could form a mental picture in space and time of the whole scheme," this is no more possible for modern physics. In Dirac's words, "it has become increasingly evident that nature works on a different plan." The fundamental laws of nature "do not govern the world as it appears in our mental picture in any very direct way": coming back to our first quotation, they control a substratum of which "we cannot form a mental picture without introducing irrelevancies" (Dirac, 1987, p. vii).

What is the value of this view in the context of the current debate on symmetries? In the next section we show how a very similar view emerges naturally when considering the general meaning of the scientific notion of symmetry.

[2] A structured picture of the current debate on physical symmetries is offered in Brading and Castellani (2003).

2. SYMMETRY, EQUIVALENCE, IRRELEVANCE

Symmetry, in its scientific sense, has developed from the initial notion of a regular arrangement of equal parts in space to the current definition in terms of invariance with respect to a transformation group. If we follow this semantic evolution in its whole course, what emerges as an essential and constant feature is the close connection that can be established between *symmetry* and *equivalence*: at the beginning, as a specific relation between symmetry and geometrical equality; in the end, as a general link between the notions of symmetry, equivalence class and transformation group.[3]

Without entering into more detail, it is in fact a general result that, given an ensemble of elements, a symmetry group corresponds to a partition of the ensemble into equivalence classes. As van Fraassen (1989, p. 234) effectively states it,

> Symmetries are transformations · · · that leave all relevant structure intact—the result is always exactly like the original, in all *relevant* respects. What the relevant respects are will differ from context to context. So settle on some respect you like: colour or height or cardinality or charm or some combination thereof. You have now partitioned your domain of discourse into *equivalence classes*.

In short, a symmetry corresponds to a situation of equivalence with respect to a given context: the elements that are like each other in all the relevant aspects are connected by symmetry transformations, so forming equivalence classes. How is this instantiated in physics? A physical symmetry corresponds to the equivalence of a number of elements—the equivalence of "identical" quantum particles, the equivalence of spacetime points, the equivalence of phase space points lying in the same gauge orbit, . . . —with respect to the physical theory considered. This means that the theory gives the same description (or the fundamental dynamical equations do not change) when the equivalent elements are exchanged with one another by the transformations of the symmetry group. This is uncontroversial. What is controversial is how to understand the equivalence of the elements.

In general, it is quite natural to think that if some elements are equivalent from the viewpoint we are considering, we do not need to take all of them into account. In some way, equivalence is linked to *irrelevance* here: the presence of equivalent elements corresponds to the presence of irrelevant features in the context considered. In physics, for example, it is quite common to understand the equivalence of space points (corresponding to translational symmetry) in the sense of the irrelevance of an absolute position to the physical description; in the same way, the equivalence of quantum particles of the same kind (permutation symmetry) is taken to signify the irrelevance of a distinction between so-called identical particles in the context of quantum theory and so on. We thus arrive very near, here, to the position expressed by Dirac: physical symmetries are related to

[3] This point is explored in detail in Castellani (2003).

the presence of irrelevant elements in the physical description; or, more precisely, in describing the physical world we introduce irrelevant theoretical elements and this is signalled by the presence of symmetries.

Note that the view has an immediate empirical counterpart in terms of non observability. The idea is that irrelevant theoretical features make no observable difference. Symmetries are thus connected with the presence of non observable quantities in the physical description, with the corollary that the empirical violation of a symmetry (the paradigmatic case is parity violation in the case of weak interactions) is intended in the sense that "what was thought to be a non observable turns out to be actually an observable."[4]

Summing up, according to the above-mentioned view—quite popular, especially among physicists—symmetries are connected with irrelevant, nonobservable theoretical features in the physical description. Now, this is still very vague: in what way are these irrelevancies introduced? What is peculiar to the ones signalled by symmetries? And, finally: is that all there is to say about the presence of physical symmetries? Thinking about the central role that symmetries play in contemporary physics, this could appear a rather minimal, if not poor, perspective.

These questions will be addressed in the following sections. The key issue in our analysis will be the *arbitrariness* that is implied in the above way of approaching the interpretative problem originated by physical symmetries. There are two kinds of arbitrariness that are connected with the presence of physical symmetries: (a) a "scale choice arbitrariness"; and (b) a "surplus structure arbitrariness."

(a) On the one side, there is a degree of arbitrariness in the distinction between what is relevant and what is irrelevant with respect to a given context. What is relevant now can be irrelevant in the future (and vice versa). And what is relevant can also depend on the level of detail at which it is considered. In physics the level of detail is usually determined in terms of the range values of a given scale (for example, an energy scale or a length scale). Here comes into play the idea which is at the basis of the recent view of current quantum field theories as *effective theories*: physics can change as one changes the scale considered, at very different ranges of energy scales we can have remarkably different physics. The form of arbitrariness involved here has thus to do with the arbitrariness of the choice of the scale range. In this paper we shall not dwell on this kind of arbitrariness.[5]

[4] Quoting from physicist Lee (1988, p. 178), a most strenuous defender of the view of symmetry in terms of nonobservability. In his words, "the root of all symmetry principles lies in the assumption that it is impossibile to observe certain basic quantities."

[5] A more detailed account of this point, exploring what can be drawn from the effective field theory approach when considering the symmetry issue, can be found in Castellani (2003).

(b) On the other side, there is always a given freedom associated with the fact that a symmetry gives rise to equivalence classes: that is, the freedom to choose one element as representative of the class. In physics, this corresponds to the presence of surplus degrees of freedom in the theory, an example of what has been called by Redhead (2003; 1975) "surplus structure." There is now a quite intensive discussion in the literature about the significance of these surplus degrees of freedom in the case of local (i.e. spacetime dependent) symmetries (Belot, 2001; Belot, and Earman, 1999; Earman, 2001, 2003; Readhead, 2003). Taking them literally leads to indeterminism; trying to "eliminate" them in some way (for example, by moving from the original phase space to a "reduced space," the points of which are equivalence classes of the original phase space points that are related by symmetry transformations) is the common stategy, but not without problems. Depending on what sort of option is chosen, there is a certain arbitrariness either in the physics (indeterminism) or in the mathematical representation (unphysical degrees of freedom giving rise to arbitrary functions of time). This situation is best dealt with by considering the relation that can be established between symmetries and *constraints*. The framework is that of the *theory of constrained systems*, going back to Dirac's seminal works in the 1950s (Dirac, 1950, 1951). Dirac's theory of constrained Hamiltonian systems and its significance for understanding the sort of arbitrariness implied by symmetries will be the subject of the next section.

3. DIRAC'S THEORY OF CONSTRAINED SYSTEMS AND THE MEANING OF GAUGES

Before turning to Dirac's Hamiltonian treatment of constrained systems, let us anticipate the reason for addressing this issue. What have constraints to do with symmetries? There is in fact an important relation between them, more precisely between gauges and constraints. As we shall see, all systems with a gauge invariance are necessarily singular systems with constraints; and, as first conjectured by Dirac in his 1950s analysis, all first-class constraints are demonstrated to be generators of gauge transformations.[6] Exploring the relevance of these results to understanding the meaning of gauges is the main aim of this paper. Note that, as current theories of known fundamental physical interactions are gauge theories, understanding gauges is without doubt a priority in today's reflections on physics and symmetries.

[6] To be precise, first-class contraints generate "small" gauge transformations, that is time (or spacetime) dependent symmetry transformations continuously connected to the identity transformation, thus defining a simply connected group. I am grateful to Jan Govaerts for letting me know the correct terminology.

Dirac's principal concepts and results concerning singular theories and constraints are best presented in his famous 1964 *Lectures on Quantum Mechanics* (Dirac, 1964). Let us then follow this text in our short overview of Dirac's analysis, from its starting point (the aim and the problem to face) to its final result.[7] Schematically

- Dirac's aim. Dirac's aim is to arrive at a general method of canonical quantization.

  To quantize, the first step is to put the classical theory in a form suitable for passing to the quantum theory, which is (for canonical quantization) the Hamiltonian form. Considering, for simplicity, the case of a system with a finite number N of degrees of freedom, with q_n $(n = 1, \ldots, N)$ the general coordinate and $\dot{q}_n = dq_n/dt$ its velocity, Dirac's method consists in[8]

  (1) starting with an action integral $S = \int L(q, \dot{q})dt$, with $L = L(q, \dot{q})$ being the Lagrange function;
  (2) getting a Lagrangian dynamics from it (i.e., deriving the equations of motion from the variation of the action integral):

  $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} = 0, \quad n = 1, \ldots, N,$$

  (3) and then passing to the Hamiltonian formulation $(q_n, \dot{q}_n) \rightarrow (q_n, p_n)$ by introducing the momentum variables p_n, which are defined by $p_n = \partial L/\partial \dot{q}_n$.
- Dirac's Problem. The above standard Hamiltonization procedure is not possible in the case that the velocities $\dot{q}_n$ are not uniquely determined in terms of the coordinates q_n and momenta p_n only; that is, in the case that the matrix $\partial p_n/\partial \dot{q}_{n'}$ is singular (its determinant vanishes). In Lagrangian terms, this is the situation expressed by the singularity of the Hessian matrix of the Lagrange function, that is by the fact that

  $$det\left(\frac{\partial^2 L}{\partial \dot{q}^n \partial \dot{q}^{n'}}\right) = 0$$

  which is the case for *singular systems*. The problem is then to put a general classical theory (including the singular case) into the Hamiltonian form.
- Dirac's solution: His Hamiltonian treatment of constrained systems. Dirac's solution is to develop a formulation of singular theories very similar to the standard Hamiltonian formulation, and then proceed to Hamiltonize. In

[7] Recent treatments of the theory of constrained Hamiltonian systems that we shall also follow here are Govaerts (1991) and Henneaux and Teitelboim (1992).

[8] For recent general treatments for all kinds of systems, see Gitman and Tyutin, 1990; Govaerts, 1991; Henneaux and Teitelboim, 1992).

substance, his strategy for dealing with the case of singular systems is grounded on the following points.

(1) *The presence of constraints*. The non invertibility of the map from velocity phase space to momentum phase space (singularity of the matrix $\partial p_n/\partial \dot{q}_{n'}$) means that not all momenta are independent functions of the velocities; that is, there must exist a set of relations of the form

$$\phi_m(q, p) = 0, \quad m = 1, \ldots, M$$

called by Dirac the "primary constraints" of the Hamiltonian formalism.[9]

(2) *The generalization of the Hamiltonian*. Given a set of N phase space degrees of freedom and a set of M primary constraints, time evolution may be generated not only by the canonical Hamiltonian $H_o = p_n \dot{q}_n - L$, but (since we may add to it any linear combination of the ϕ's which are zero) by a generalized Hamiltonian of the form

$$H_* = H_o + u_m(q_n, p_n; t)\phi_m(q_n, p_n)$$

where ϕ_m are all primary constraints and u_m are arbitrary functions of time and of the phase space variables.[10]

Starting with this expression and carrying out his analysis by imposing all the consistency requirements of the theory, Dirac arrives at the following final expression for the generalized Hamiltonian:

$$H_* = H + v_a(t)\phi_a,$$

where $H = H_o + U_m\phi_m$ and $\phi_a = V_{am}\phi_m$. The U_m and V_{am} are functions of the phase-space variables satisfying to given consistency equations, while the $v_a(t)$ are *arbitrary functions of time* (their number being equal to the number of primary first-class constraints, usually less than the number of all constraints).[11]

(3) *The arbitrariness implied and its consequences*. The result is that, although all consistency requirements have been satisfied, the theory has still arbitrary coefficients which may depend on time, the $v_a(t)$. This means that the general solution of the Hamiltonian equations of motion, with given initial conditions, depends on arbirary functions of

[9] Dirac distinguishes between *primary* and *secondary* constraints (depending on whether their definition is independent of the Lagrangian equations of motion or not), and *first-class* and *second-class* constraints (depending on whether their Poisson brackets with all other constraints is weakly vanishing or not). The important distinction for the treatment of constrained systems is the second one. For details see Dirac (1964), pp. 14–18.

[10] We follow here, in part, also the notation used in Govaerts (1991).

[11] For details, see Dirac (1964, pp. 13–16) and Govaerts (1991, pp. 91–95).

time, which is due to the fact that the description of singular systems includes dependent degrees of freedom. Quoting Dirac (1964, p. 17):

> These arbitrary functions of the time must mean that we are using a mathematical framework containing arbitrary features, for example a coordinate system which we can choose in some arbitrary way, or the gauge in electrodynamics. As a result of this arbitrariness in the mathematical framework, the dynamical variables at future times are not completely determined by the initial dynamical variables, and this shows itself up through arbitrary functions appearing in the general solution.

What the arbitrariness exactly consists in and where the origin of the surplus theoretical features is (how they enter in the theory) is made here quite precise. Namely, as Dirac clearly states, the arbitrariness in the choice of the functions $v_a(t)$ implies that the different trajectories in phase space obtained under time evolution for given initial conditions but for different $v_a(t)$ should be considered as representing the same configuration of the system. How are these different points and trajectories in phase space representing the same time dependent state of the system related to one another? At this point, the actual meaning of first-class constraints as generators of local Hamiltonian gauge transformations, which is the upshot of the final part of Dirac's analysis, comes into play.

(4) *The meaning of first-class constraints*. To get a physical understanding of the above situation—we start with given initial variables q's and p's and get a solution of the equations of motion containing arbitrary functions—Dirac proceeds as follows:

(a) As the q_n and the p_n at later times are not uniquely determined by the initial state, because of the arbitrary functions coming in, the problem is to look for all the sets of q_n and p_n that correspond to one particular physical state. All those values for the q_n and p_n at a certain time which can evolve from one initial state must correspond to the same physical state at that time.

(b) Taking then particular initial values for the q_n and p_n at time $t = 0$, and considering what the q_n and p_n are after a short time interval δt, the value at time δt of a general dynamical variable g is

$$\begin{aligned} g(\delta t) &= g_o + \dot{g}\delta t \\ &= g_o + [g, H_*]\delta t \\ &= g_o + \delta t\{[g, H] + v_a[g, \phi_a]\} \end{aligned}$$

(c) As the coefficients v_a are arbitrary, we can take different values v'_a.

This gives a different $g(\delta t)$, the difference being:

$$\Delta g(\delta t) = \delta t(v_a - v'_a)[g, \phi_a],$$

or

$$\Delta g(\delta t) = \epsilon_a[g, \phi_a],$$

where

$$\epsilon_a = \delta t(v_a - v'_a).$$

We can change all Hamiltonian variables in accordance with the rule $\Delta g(\delta t) = \epsilon_a[g, \phi_a]$, and the new Hamiltonian variables will describe the same state. Now, Dirac notes, this change in the Hamiltonian variables consists in applying an infinitesimal contact transformation with a generating function $\epsilon_a \phi_a$ (that is, a gauge transformation). By then considering also the case of secondary constraints, Dirac (1964, p. 23) finally arrives at the following conclusion:

> The final result is that those transformations of the dynamical variables which do not change physical states are infinitesimal contact transformations in which the generating function is a primary first-class constraint or possibly a secondary first-class constraint.

Dirac's conjecture that his result should regard also secondary first-class constraints was indeed correct, as it has been later demonstrated.[12] The upshot of Dirac's theory is then, in general terms, that *all first-class constraints are generators of gauge transformations*. More precisely, first-class constraints generate transformations which relate the different phase space points (all taken at equal time) that belong to different trajectories corresponding to the same configuration of the system. These are infinitesimal transformations leaving the Hamiltonian description of the system invariant, i.e. the gauge symmetries of the system.[13]

Now, what does this tell us about the meaning of gauges? Let us first shortly consider the other side of the connection between gauge and constraints, that is the fact that gauge theories are singular theories with constraints. According to Noether's second theorem, we have that for a gauge invariant system—that is, a system invariant under transformations defining a simply connected continuous group, whose parameters are r arbitrary functions of time (or spacetime)—there are r independent identities of the Euler–Lagrange derivates of the Lagrange function. These identities are consequences of the gauge invariance and define (Lagrangian) constraints on the system. That gauge invariant systems are singular systems may be easily seen by analyzing these identities. We thus arrive at the result that all

[12] The result, obtained in Castellani (1982), is actually valid in a slightly different form with respect to the one conjectured by Dirac. But we do not need to enter into such details here.

[13] See for details Govaerts (1991), 116–121.

systems with a gauge invariance are indeed singular systems with constraints, with the related presence of arbitrary functions of time in the general solutions of the equations of motion.[14]

Note that the above result is not at all surprising: gauge transformations do in fact naturally involve arbitrary functions of time. Quoting Henneaux and Teitelboim (1992, p. 3), "a gauge theory may be thought of as one in which the dynamical variables are specified with respect to a 'reference frame' whose choice is arbitrary at every instant of time." Which means that one cannot expect that the equations of motion will determine all the dynamical variables for all times for specific initial conditions (it will always be possible to change the reference frame at a future time, while keeping the initial conditions fixed). The actual meaning of these arbitrary functions of time and what they in fact imply is then made clear by analyzing gauge systems as constrained Hamiltonian systems.

4. CONCLUSION

We are faced with situations where a physical system is described by more variables than there are physically independent degrees of freedom. We have seen above in what the arbitrariness peculiar to this sort of situation consists, and how the connected surplus theoretical features enter into the theory. In particular: many different phase space points and trajectories representing the same time dependent configuration of the system. The Hamiltonian treatment à la Dirac shows how they are related to one another by gauge transformations. How should we deal, then, with these equivalent or redundant descriptions? The most natural way is by keeping the physically meaningful degrees of freedom only (ideally, by means of a description of the dynamical evolution of the system in terms of the "reduced" space representing the truly distinct possible configurations of the system). In gauge terms, this corresponds to the common strategy of attributing physical meaning to gauge invariant quantities only. The underlying idea is that in a theory where the system is described by more variables than the number of independent degrees of freedom, "the physically meaningful degrees of freedom reemerge as being those invariant under a transformation connecting the variables (gauge transformation)" (Henneaux and Teitelboim, 1992, p. 1).

The philosophy, in substance, is the following: for given reasons we introduce extra variables (surplus theoretical features or "irrelevancies," in Dirac's words) in the theory, and at the same time we bring in a (gauge) symmetry "to extract the physically relevant context" (Henneaux and Teitelboim, 1992, p. 1). The view explored in Section 2, according to which we introduce irrelevant theoretical elements in describing the physical world and this is signalled by the presence of

[14] For details on this part and, especially, on the exact relation between the Lagrangian and Hamiltonian treatments, see Gitman and Tyutin (1990) and Govaerts (1991).

symmetries, is thus made quite precise. We can now complete our initial quotation from Dirac's 1930 preface (1987, p. vii):

> This state of affairs is very satisfactory from a philosophical point of view, as implying an increasing recognition of the part played by the observer in himself introducing the regularities that appear in his observations, and a lack of arbitrariness in the ways of nature . . . .

To conclude, what we have sketched in this paper—following its historical and conceptual roots—is a strong epistemic attitude in considering the meaning of symmetries. Now, it is our opinion that it is possible to maintain the basic motivations for such a view, but in a "milder" way. The idea, that we develop elsewhere,[15] is a sort of compromise: symmetries do enter in our way of describing the physical world, but in a way that is significantly "constrained" by the reality we want to describe. In this regard, particular relevance is attributed to the role played by boundaries conditions and physical scales. But for this we have to leave Dirac and follow another path.

ACKNOWLEDGMENT

I am very grateful to Katherine Brading, Leonardo Castellani, Marisa Dalla Chiara, Jan Govaerts, and Josep Pons for helpful comments and suggestions.

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